MINIMAL SURFACES IN HYPERBOLIC GEOMETRY

Winter School Côte d'Azur

Lecture II, 7th January 2026

Goal: sketch a proof of the following

Theorem (Huang - Wang '15, '19)

For every $g \geq 2$ and $n \geq 1$, there exists a quasi-Fuchsian manifold (M, h), $M \cong S_g \times \mathbb{R}$, containing at least n closed minimal surfaces homotopic to $S_g \times \{*\}$.

Ideas here based on Huang - Wang '19, Anderson, Farre - Vargas Pallete.

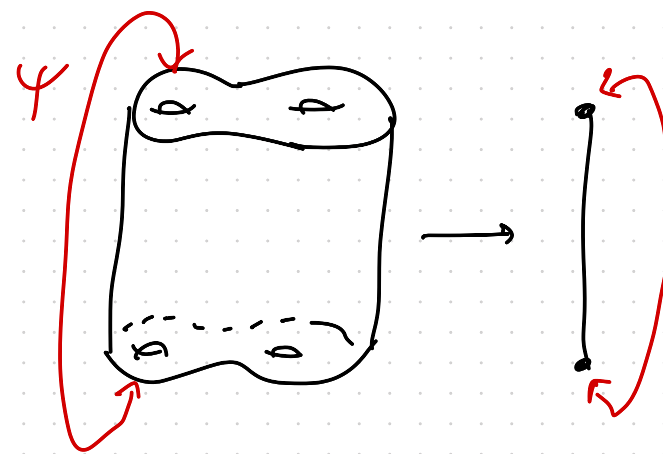
A mapping torus is a 3-manifold of the form

$$N_\varphi = S_g \times [0,1] /_\sim$$

where $(p,0) \sim (\varphi(p), 1)$ for $\varphi : S_g \to S_g$ orientation-preserving diffeo.

The map $[(p,t)] \longmapsto t$ induces a fibration

$$N_\varphi \longrightarrow S^1 = [0,1]/_\sim$$



<u>Theorem</u> Given $\varphi : S_g \to S_g$ diffeo

- φ has finite order $\iff$ N_φ has geometry $\mathbb{H}^2 \times \mathbb{R}$
- φ is reducible $\iff$ N_φ has an essential torus
- φ is pseudo-Anosov $\iff$ N_φ is hyperbolic.

Nielsen - Thurston classification of surface diffeomorphisms (up to $\mathrm{Diff}_0(S_g)$)

geometrization

So, if φ is pseudo-Anosov, then N_φ is a closed hyperbolic manifold.

<u>Theorem</u> (Virtually Fibered Conjecture, Wise + Agol 2009-2012)

Every closed hyperbolic 3-manifold has a finite cover that is a pseudo-Anosov mapping torus.

Now, let $\psi : S_g \to S_g$ pseudo-Anosov.

We have

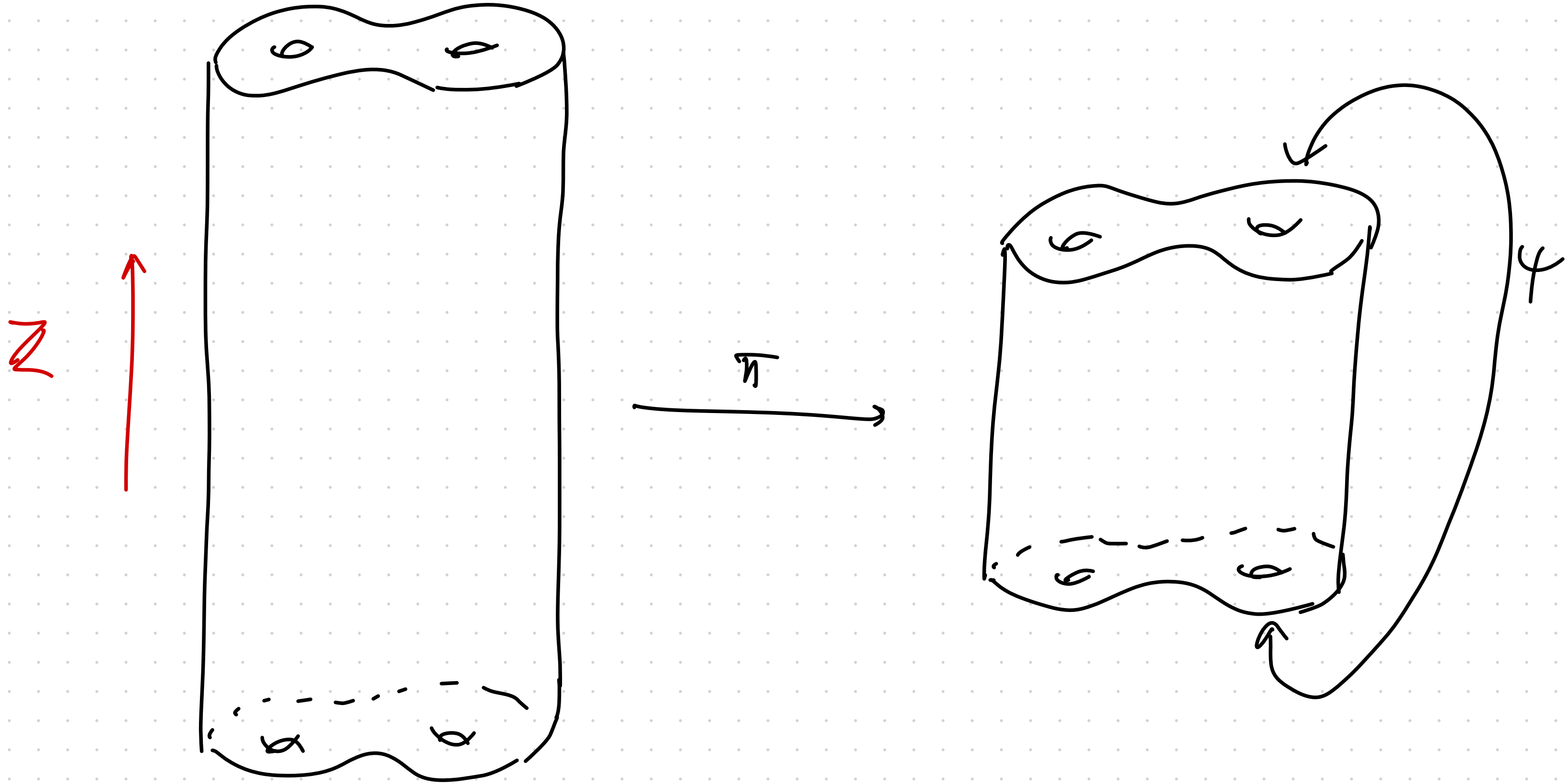
$$1 \longrightarrow \pi_1 S_g \xrightarrow{\iota} \pi_1 N_\psi \longrightarrow \pi_1 S^1 \cong \mathbb{Z} \longrightarrow 1$$

$$\Gamma = \iota_*(\pi_1 S_g)$$

Let $M_\psi = \hat{N}_\psi \xrightarrow{\pi} N_\psi$ the covering of N_ψ such that $\pi_*(\pi_1 M) = \Gamma$

$$\text{Then } [(M_\psi, \overset{\pi^* h}{\overset{\shortparallel}{h_\psi}})] \in AH(S_g)$$

Actually, it is in $\partial AH(S_g)$



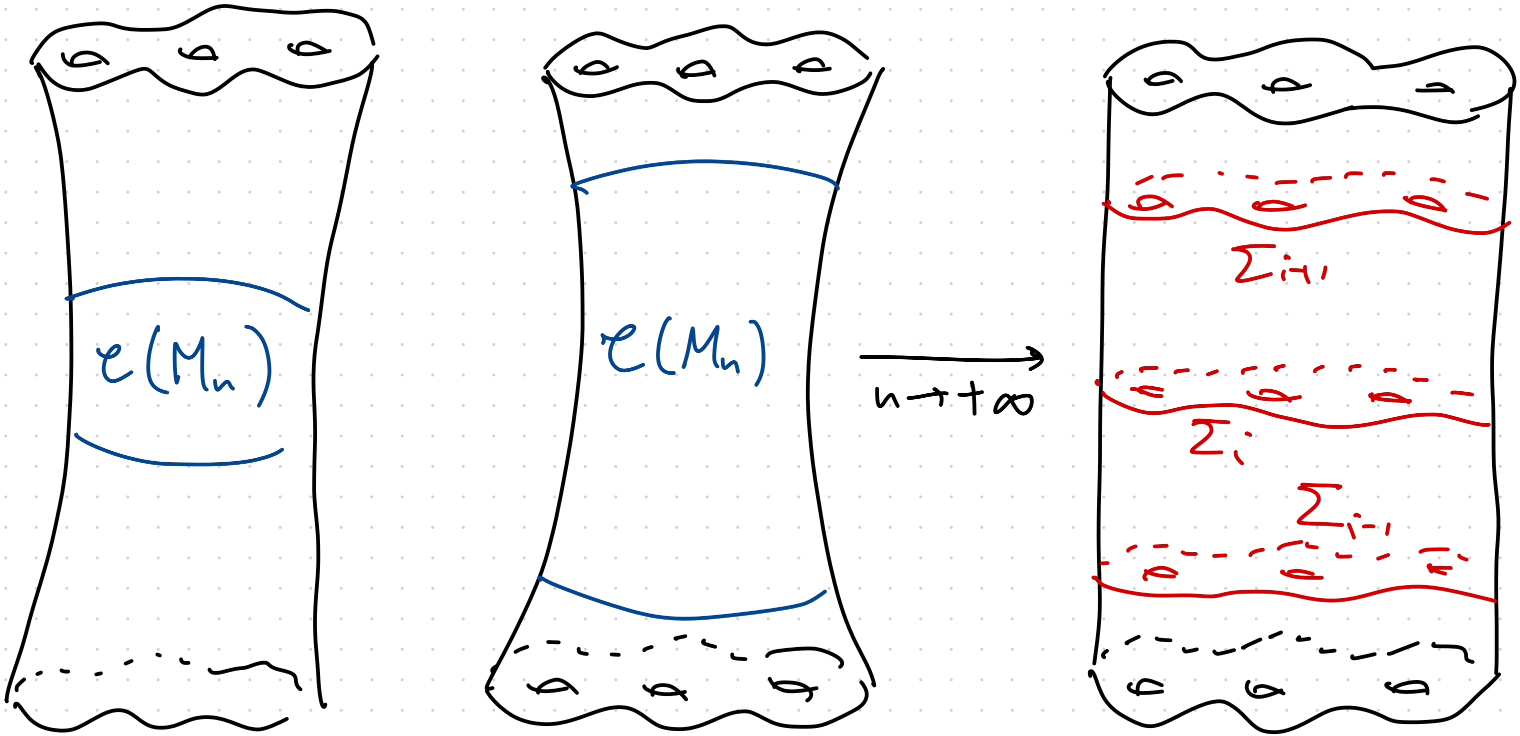
(M_ψ, h_ψ)

$\mathcal{C}(M_\psi) = M_\psi \implies M_\psi$ not quasi-Fuchsian.

By general results (Schoen-Yau, Sachs-Uhlenbeck), N_φ contains an area minimizing minimal surface homotopic to $S_g \xhookrightarrow[\text{fiber}]{} N_\varphi$

$\rightsquigarrow$ M_φ has infinitely many closed minimal surfaces Σ_i of genus g that minimize area in a neighbourhood of Σ_i $(i \in \mathbb{Z})$.

Now, let (M_n, h_n) a sequence of quasi-Fuchsian manifolds converging to (M_ψ, h_ψ) in $AH(S_g)$.



Rough idea: each minimal surface Σ_i should be a limit of minimal surfaces in M_n

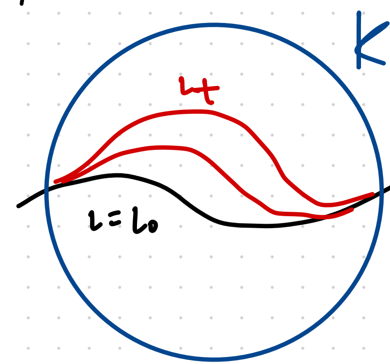
$\rightsquigarrow$ M_n has an arbitrarily large number of minimal surfaces for n big.

This works well if Σ_i are strictly stable

First variation formula:

if $\iota_t(p) = \exp_{\iota(p)}(t f(p) N(p))$, $f \in C_0^\infty(S)$, then

$$\frac{d}{dt}\Big|_{t=0} \operatorname{Area}(\iota_t(S)) = -\int_S f H \, dA_I$$



so, ι is critical point of area $\Leftrightarrow H \equiv 0$.

Second variation formula

$\lambda^2 + \mu^2$

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Area}(\iota_t(S)) = \int_S \left(\|\nabla f\|^2 - (\|\mathrm{II}_\Sigma\|^2 - 2) f \right) dA_\Sigma$$

≥ 0 if Σ WAF

$$= - \int_S f \, L_\Sigma(f) \, dA_\Sigma$$

for $L_\Sigma(f) = -\Delta_\Sigma f - (\|\mathrm{II}_\Sigma\|^2 - 2) f$

Morally, L_Σ is the derivative of the mean curvature

$\Longrightarrow$ if L_Σ is invertible, then one can use the implicit function theorem.

Problem: for an area-minimizing surface,

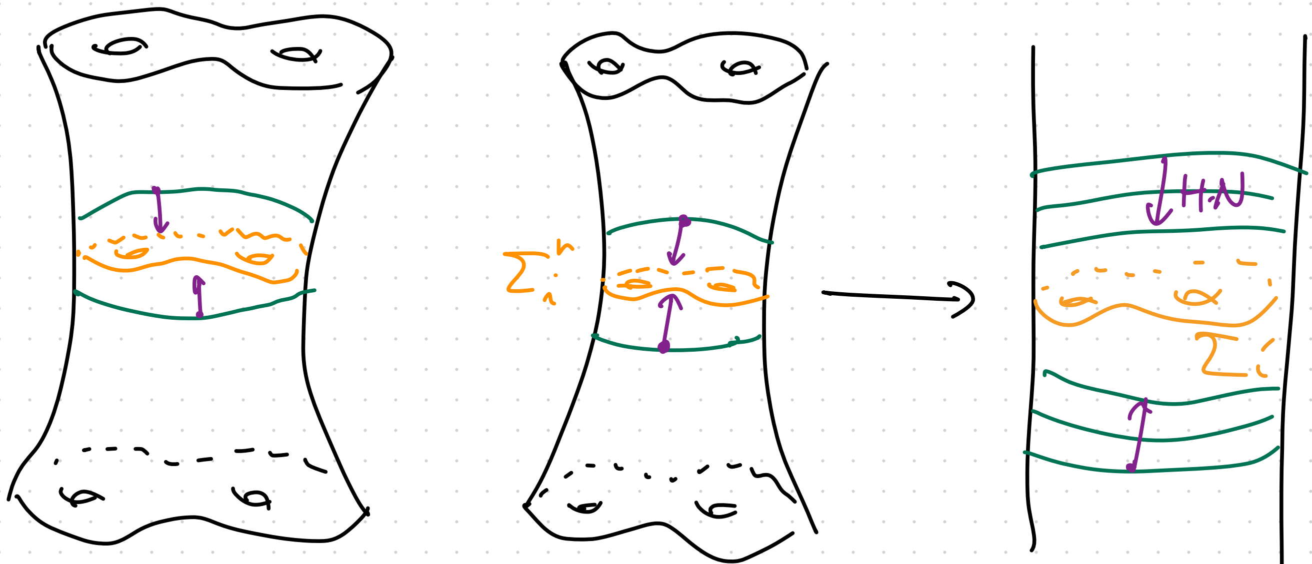
$$\frac{d}{dt^2}\Big|_{t=0} \text{Area} \geq 0 \implies \int_\Sigma f L_\Sigma(f)\, dA_\Sigma \geq 0$$

but $L_\Sigma(f)$ might have kernel

("stable" but not "strictly stable")

<u>Solution</u>: Σ_i has a neighbourhood foliated by surfaces where the sign of the mean curvature is constant on each side ($H>0$, $H<0$ or $H=0$)

- if $H>0$, one can find minimal surfaces in the sequence M_n for n large!



- $H < 0$ not possible if Σ_i is area-minimizing
- $H = 0$?

<u>Conjecture</u> No hyperbolic three-manifold has a (local) foliation by closed minimal surfaces

<u>Anderson + Huang-Wang</u>: there exist some hyperbolic mapping torus N_ψ that has no local foliation by closed minimal surfaces!